DOCUMENT RESUME

ED 436 255 PS 028 091

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TITLE	Fostering Each Child's Understanding of Mathematics.
SPONS AGENCY	National Science Foundation, Arlington, VA.
PUB DATE	1999-00-00
NOTE	28p.; In: The Early Childhood Curriculum: Current Findings in Theory and Practice. Third Edition; see PS 028 086.
PUB TYPE	Information Analyses (070)
EDRS PRICE	EDRS Price MF01 Plus Postage. PC Not Available from EDRS.
DESCRIPTORS	Cognitive Development; Curriculum Development; *Elementary Education; *Elementary School Curriculum; Elementary School Students; Mathematical Concepts; *Mathematics Education; Mathematics Instruction; National Standards; Number Concepts; Numeracy; Problem Solving; Spatial Ability

ABSTRACT

Noting that assumptions about the teaching and learning of early childhood mathematics have changed substantially since the 1970s, this chapter examines recent research that may inform curriculum development in early childhood mathematics. The chapter begins with a consideration of the theoretical perspectives for defining an appropriate mathematics curriculum, focusing on mathematics as a dynamic discipline, the construction of knowledge, perceiving relationships between or within mathematical ideas, and the role of social interaction in learning mathematics. The chapter then interprets research on children's mathematical thinking, including the development of number sense, the concept of quantity, part-whole constructions, counting and numeration, place-value instruction, problem-solving abilities, representation, geometric concepts, and spatial sense. Finally, the chapter reviews instructional issues that must be addressed when creating learning environments, drawing from current research that considers classroom discourse, teacher questioning, and instructional decision making. (Contains approximately 90 references.) (KB)

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CHAPTER 5

Fostering Each Child's Understanding of Mathematics

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EXAMINATION OF early childhood mathematics curricula since the mid-1970s reveals a shift from viewing mathematics as a set of skills and procedures, to defining problem solving as "the focus of school mathematics" (National Council of Teachers of Mathematics [NCTM], 1980, p. 1), to characterizing mathematics as the "science of pattern and order" (National Research Council, 1989, p. 31). In recent curricula, mathematics is viewed as a way of thinking about quantity, relationships, and patterns through modeling, symbolism, inference, analysis, and abstraction. Problem solving in mathematics is no longer limited to end-of-the-unit word problems. Rather, preschool and primary-aged children are challenged to make sense of the mathematics in problem situations that arise in their direct experience, to make sense of symbolic mathematics as recordings of meaningful conceptual relationships, and to make sense of observable characteristics leading to generalizable geometric properties and mathematical patterns. This does not mean that computation, traditional word problems, and numerical algorithms are no longer important. It does mean that assumptions about the teaching and learning of early childhood mathematics have changed substantially since the 1970s.

Reports guiding curriculum reform in mathematics education (e.g., Commission on Standards for School Mathematics, 1989; Commission on Teaching Standards for School Mathematics, 1991; National Research Council, 1989) were informed by a substantial body of research developed over the past 15 years about how children learn mathematics, par-

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ticularly about how children come to understand number and quantitative relations. The major assumption underlying the current reform in mathematics education is the perception that students should understand mathematics (Hiebert & Carpenter, 1992). While this idea is not uniformly accepted by the public at large, it is one of the most widely held tenets in the mathematics education community.

This chapter examines recent research that may inform curriculum development in early childhood mathematics. Theoretical perspectives for defining an appropriate mathematics curriculum are considered first. Second, because knowledge about children's thinking can provide a framework for defining curriculum (Carpenter, 1988), interpretations of research on children's mathematical thinking are presented. The chapter then reviews instructional issues that must be addressed when creating learning environments, drawing from current research projects that consider classroom discourse, teacher questioning, and instructional decision making.

THEORETICAL PERSPECTIVES ON TEACHING AND LEARNING MATHEMATICS

In 1989, two documents were published that presented a vision of the mathematics curriculum as a broad range of content, encompassing a variety of applied and abstract contexts, with deliberate connections being drawn between mathematical ideas (Commission on Standards for School Mathematics, 1989; National Research Council, 1989). Furthermore, this perspective called for instruction wherein children communicated and reasoned about their interpretation of the mathematics. There are those who promote an opposing view of school mathematics, interpreting algorithmic procedures as the foundation of school mathematics, with demonstration and practice framing instruction because skill mastery is a presumed prerequisite for conceptual knowledge. Yet, within the mathematics education community, there seem to be four themes regarding the nature of mathematics and mathematical knowledge:

1. Mathematics is a growing, dynamic discipline.
2. Students actively construct mathematical knowledge.
3. Understanding in mathematics comes from perceiving relationships either between or within mathematical ideas.
4. Knowledge may be fostered through social interaction.

Mathematics—A Dynamic Discipline

"Knowledge is not a basket of facts" (Anderson, 1984, p. 5), and mathematics is not simply a static network of terms, rules, and procedures that are conveyed by teachers and absorbed by students for recall upon demand. Mathematics is the active science of defining relationships in the search for order and pattern in the world around us (National Research Council, 1989). Defining relationships implies that a mathematical topic or situation is perceived as a component of, or connected to, a "big idea" (Campbell & Johnson, 1995). In early childhood education, a critical aspect of curriculum development is to select and define those big ideas, for it is through the process of constructing meaning for the big ideas that children come to understand mathematics. Over time, as children apply their existing understandings to new mathematical experiences and problems, children not only learn new content, their "individual knowledge of an existing portion of mathematics" grows (Dossey, 1992, p. 44).

The challenge in early childhood mathematics education is "to use children's intial understandings to proactively support their *continued* mathematical development" (Campbell, 1997, p. 106). That is, in early childhood mathematics, the content of the curriculum and the instructional approach must challenge without frustration so that children will build "progressively more advanced understandings and simultaneously perceive mathematics as 'making sense'" (p. 106). This means that children must be engaged in instructional activity that causes them to explore, examine, quantify, order, relate, invent, use, represent, verify, interpret, justify, and communicate (Commission on Standards for School Mathematics, 1989). It also means that the underlying goal is for children to construct understandings that are consistent with established standards for mathematics. Mathematics curricula and instruction must emphasize the power and potential of mathematics, the interrelatedness of mathematical ideas, and the development of quantitative reasoning, as well as each child's individual understandings and confidence to undertake such investigation (Romberg & Tufte, 1987).

Construction of Knowledge

This perception of mathematics and of a mathematics curriculum does not coincide with instruction wherein bits of information that compose a network of established rules and procedures are transmitted to children (via either textbooks, worksheets, or teacher demonstration). One current theoretical perspective is that students actively construct mathematical knowledge; they do not absorb mathematics or receive it. The basic

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tenet is that knowledge is constructed by actively processing and relating new information or experiences to what was already known or by connecting ideas that were previously understood but isolated from each other (Noddings, 1990). Thus learning is more than simply accumulating bits of information. Learning means new insights in the way that one thinks about something. However, these insights are interpreted from within; they are not meaningless responses in a manner previously prescribed by an authority figure (cf. Baroody, 1987). This perspective, generally termed *constructivism,* is rooted in the writings of Piaget. While there are philosophical and epistemological distinctions in the implications drawn by constructivist theorists (e.g., von Glasersfeld, 1991), the perspective that knowledge is actively constructed rather than passively received has had critical implications for mathematics education.

In their studies of mathematical knowledge, researchers distinguish between conceptual knowledge and procedural knowledge. Conceptual knowledge refers to the knowledge of relationships. Mathematical concepts—that is, principles and constraints that define or characterize ideas and experiences—are relationships; yet these concepts can be related or integrated with other concepts. Children develop conceptual knowledge when they construct relationships between pieces of information. Procedural knowledge entails (1) the written symbols that are used to represent mathematical ideas and (2) the rules and procedures that are used when solving mathematics problems. These procedures should entail strategies that foster the solution of problems while preserving or enriching the constructed meaning of mathematical concepts. Thus procedural knowledge should have a conceptual base and generally has a symbolic referent (Gelman & Greeno, 1989; Hiebert & Lefevre, 1986).

Mathematics Instruction

As noted by Pierie and Kieren (1992), there is "no 'constructivist teaching model' out there waiting to be implemented" (p. 506). However, because the constructivist view of mathematics learning raises implications for instruction, there have been numerous studies to investigate primary mathematics instruction based on a constructivist model. These studies generally apply one of two instructional approaches, each of which relies on research-based knowledge of the development of children's learning of mathematics but differs in what it defines to be the optimal learning environment.

One approach uses visible representations (manipulative materials, diagrams, and/or symbols) to characterize the desired mathematical relationship or concept in a form that is felt to be more meaningful or access-

ible to the child. The critical points of the instruction revolve around making connections between the varying representations (e.g., between objects and numerals) as a means of clarifying the mathematical concept (Fuson & Briars, 1990; Hiebert & Wearne, 1992; Resnick, 1983a, 1983b). The level of direction provided by the teacher regarding manipulation of the symbolic or concrete referents varies in these studies, but it is often quite directive during the initial lessons. A question raised regarding this instructional approach is what meaning is attached to these visible representations by the child (Baroody, 1989; Cobb et al., 1991). The issue is as follows: The teacher or adult readily identifies the intended mathematical structure within the referent. But is that referent meaningful to the child? Does the child interpret the same mathematical structure perceived by the adult, some variant of that structure, or no structure at all (Nesher, 1989)? Further, if a teacher directly instructs or demonstrates a procedure, either symbolically or with manipulatives, does that limit the probability that the child will reason and make sense of the activity? Might the child simply attempt to mimic the teacher's actions (Cobb, 1988)? Questions regarding children's mathematical sense making are addressed later in the chapter in terms of children's use of place-value manipulatives.

Another approach being studied also invoives the use of manipulatives but limits teachers' direct instruction. This approach uses cognitive-based research as a means of identifying mathematical activities wherein children are engaged in solving problems that are challenging but conceptually attainable (Campbell & White, 1997; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Carpenter, Franke, Jacobs, Fennema, & Empson, 1998; Cobb et al., 1991; Fuson, Wearne, et al., 1997; Wood & Sellers, 1997). These activities may also be intended to provoke the child to relate existing understandings, to construct new knowledge, or to develop strategies or procedures. Because circumstantially based knowledge may be applied by a child when solving problems based in a context that is familiar to that child (Leinhardt, 1988), these instructional activities may evolve from experiences common and familiar to the children. In this approach, the teacher does not demand a particular procedure or expect the children to use their materials in a prescribed manner; no strategy is valued more than another. Children are given the opportunity to solve the problems; frequently strategies are constructed within the social construct of the classroom as children share their ideas and invented strategies. This approach is not simply discovery learning. Rather, the teacher must pose tasks that may stimulate appropriate conceptual reorganizations in the students. This then requires the teacher to understand the expected developmental sequence of mathematical understandings, as

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well as the current levels of mathematical understandings held by the children. Further, teachers must maintain standards for what is mathematically correct, generalizable, and efficient, raising questions to promote each child's mathematical growth without devaluing any child's current level of understanding (Campbell, 1997).

Researchers in mathematics education also differ regarding the use of manipulative materials within this approach. While some researchers (e.g., Cobb et al., 1991; Fennema, Carpenter, & Peterson, 1989; Fuson, Wearne, et al., 1997) use manipulatives, they do so after first considering the degree to which that referent might foster each child's own thinking and the potential for that referent to assist the children in communicating their meaning to each other and to the teacher (Fischbein, 1987). However, Kamii (1990; Kamii & Joseph, 1988) opposes the use of manipulatives such as bundles of counting sticks, saying that young children neither abstract concepts from concrete referents nor internalize concepts from the external, physical environment.

Social Interaction and the Learning of Mathematics

Social-cognitive and social-constructivist theories posit the importance of student interaction in the mathematics classroom (Cobb, Yackel, & Wood, 1992). When children share their interpretations or conjectures, offer verification or explain their reasoning, question each other, and, if necessary, expand upon or justify their perceptions, there is an increased likelihood that they will correct, interpret, and reorganize their own thinking (Ball, 1993; Lampert, 1990; Resnick, 1988). Cobb and his colleagues (1991) hypothesize that classrooms that foster this type of social interaction permit the children and the teacher to "mutually construct taken-to-be-shared mathematical interpretations and understandings" (p. 6). A number of research studies have investigated the teaching and learning of mathematics within classrooms that seek to employ these social settings, noting higher levels of perfomance, particularly in mathematical problem solving (e.g., Campbell & Rowan, 1995; Carpenter et al., 1989; Cobb et al., 1991; Hiebert & Wearne, 1993; Wood & Sellers, 1997).

A common characteristic of the social interaction in many of these classrooms is the sharing of approaches or strategies by the children without a labeling or a judgment of "correct" or "incorrect" by the teacher. This does not mean that all answers are interpreted as being correct. Learning will not be fostered if teachers accept all responses without further interaction simply because the answer is an expression of a child's understanding. Rather, through questioning and through individual, group, or class interaction with the teacher, the children determine and

justify the mathematical validity of answers. Teacher and student questioning may cause a child to address a misconception or to experience dissonance and to subsequently modify his or her thinking. At other times, even though questioning may not cause a child's understanding to change to reflect adult society's perception of mathematics, the child's thinking will become more explict to the teacher. Then the teacher must reflect on what kinds of problems or experiences might be necessary to further stimulate that child's mathematical growth (Campbell & Johnson, 1995). In any case, however, the climate in which the questioning and interaction take place must enhance children's trust and willingness to talk about mathematics and support their confidence to investigate further problems.

CHILDREN'S MATHEMATICAL THINKING

Developing Number Sense

"The major objective of elementary school mathematics should be to develop number sense" (National Research Council, 1989, p. 46). This recommendation identifies a critical distinction between traditional mathematics programs for young children and a mathematics curriculum that supports and fosters children's understanding. Number sense refers to a child's "intuition about numbers, their magnitudes, their effects in operations, and their relationships to real quantities and phenomena" (Van de Walle, 1998, p. 4). The NCTM *Curriculum and Evaluation Standards for School Mathematics* (Commission on Standards for School Mathematics, 1989) notes that children who have number sense understand the meaning of number as they are able to define many different relationships among numbers. In addition to recognizing the relative size of numbers and the effect of operating on numbers, these children define their own referents for use in measuring common objects and events. Number sense is a flexible way of thinking about numbers, and it presupposes a child's belief that mathematics should and does make sense (Carpenter, 1989; Resnick, 1989; Silver, 1989). Because it entails flexible thinking about numerical situations, it is presumed that number sense advances problem solving, supports children's construction of more enriched understandings, and perpetuates its own development.

According to a cognitivist perspective, if a child has a concept of a number, then the child has constructed a meaning for that number in terms of its relationship to other numbers. For example, 5 is

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2 and 3
1 more than 4
A lot less than 30
Only a little less than 7
As many fingers as are on one hand
With 5 more, 10 altogether

An early childhood curriculum that fosters the development of number sense permits children to understand number as more than the product of counting.

Early Constructs of Quantity. The next section of this chapter addresses research regarding the development of meaningful counting, from rote counting to place-value numeration concepts. But first, consider the development of a concept of number. Through experiences in the home, young children develop primitive notions of quantity. As children begin to solve problems in their environment involving quantity, they begin to construct meaning for number (Fuson, 1989; Gelman & Meck, 1986, 1992; Saxe, Guberman, & Gearhart, 1987). Steffe and his colleagues (Steffe, Cobb, & von Glasersfeld, 1988) have investigated the types of entities or settings for which children display increasingly sophisticated meanings for number. They note that initially children count collections of objects and learn to construct sets of a given size but require physical objects to which they attach their numerical words. With growth, children begin to quantify the motions they create (pointing to locations, putting up fingers). Eventually, children abstract counting. When a child abstracts counting, the child's construction of meaning for a number (e.g., 5) is independent of the presence of concrete objects or motor activity. To a child who is an abstract counter, number refers not only to any set, image, or action that contains or depicts that many items but also to the sequence bounded by that number (e.g., 1, 2, 3, 4, 5). Thus abstract counters can count the act of counting. For example, in the problem, *Marissa has 7 stickers. How many more stickers does she need to have 11 stickers?* an abstract counter may respond, "8, 9, 10, 11. . . . She needs 4 more stickers."

Part-Whole Constructions. A number may be associated with a set of objects. However, that set can also be separated into subsets, and each of those subsets also has a referent number. This critical partitioning relationship within number is generally termed part–part–whole. Resnick (1983a) noted the importance of this construction, stating:

> Probably the major conceptual achievement of the early school years is the interpretation of numbers in terms of part and whole relationships. With the application of a Part–Whole schema to quantity, it becomes possible for children to think about numbers as compositions of other numbers. This enrichment of number understanding permits forms of mathematical problem solving and interpretation that are not available to younger children. (p. 114)

With a part–whole construction, children may relate the concepts of number, addition, and subtraction (Baroody, Ginsburg, & Waxman, 1983; Campbell & Johnson, 1995); reach the most advanced problem-solving level for addition and subtraction problems (Riley, Greeno, & Heller, 1983); and support the conceptual adjustments involved when learning place value (Resnick, 1983a). Fischer (1990) noted that a kindergarten mathematics curriculum based on a part–part–whole approach fostered children's development of number concepts and facilitated their construction of strategies to solve addition and subtraction word problems, even though addition and subtraction applications were not taught.

Counting and Numeration

Early Counting. The ability to find out *how many* is a powerful tool that children use as they begin to model and solve problems. Researchers offer differing perspectives regarding the meaning and growth of counting. Some researchers contend that children first learn a rote sequence of number words without reference to the objects being counted and then make the transition to coordinating each number word in the sequence with an object in a set to determine the cardinality of the set (Baroody, 1993; Frye, Braisby, Lowe, Maroudas, & Nicholls, 1989; Fuson, 1989, 1992; Sophian, 1992; Wynn, 1990). These researchers do not presume that initial "attempt[s] to carry out counting activity" (Fuson, 1989, p. 398) indicate a manifestation of implicit counting principles. Others suggest that even very young children have some implicit principled understanding of counting because they respond to the presence of quantity in their environment (Gelman & Greeno, 1989; Gelman & Meck, 1986, 1992; Strauss & Curtis, 1984). Despite differences in defining an initial principled knowledge of counting, it is generally accepted that explicit knowledge of counting is a prerequisite for increasing, decreasing, and comparing quantities (Greeno, Riley, & Gelman, 1984). In most instances, the ability to count rationally enables children to solve simple word problems.

Gelman and Gallistel (1978) define five principles of counting:

1. *Stable order*—Counting names are used in a stable order.
2. *One–one correspondence*—Every object is assigned one and only one unique counting name.
3. *Cardinality*—The last counting name identifies the number in the set.
4. *Order irrelevance*—Objects may be counted in any order and a change in the order does not affect the total.
5. *Abstraction*—Any set, or collection, of objects may be counted.

Gelman and Meck (1986) posit that the first three principles represent a developmental sequence in learning how to count. Fuson (1992) characterizes five levels of qualitative change in children's use of number words within a counting task. For example, initially children must begin their counting sequence with the words, "1, 2, 3, . . ." As their understanding of number matures, they are able to begin a count with a number other than 1.

Numeration. Children initially use number words to represent *unitary* conceptual structures, where each number word refers to a single item in a collection of objects (Fuson, 1990; see also Fuson, Smith, et al., 1997; Fuson, Wearne, et al., 1997). For example, a child counting a set of 26 blocks will count "1, 2, 3, . . . 24, 25, 26" and will interpret the number word "twenty-six" as referring to the total collection of 26 single objects. This interpretation of number does not reflect an understanding of our base-ten number system but does allow a child to operate with numbers larger than ten. A more sophisticated structure for multidigit numbers is constructed by children when they "disembed" groups of 10 objects from the total collection, recognizing that an item may simultaneously be assigned to a group (1 of 10 objects) and to the collection (1 of 26 objects) (Steffe et al., 1988). When children disembed, they have begun "to construct *multiunit* conceptual structures in which the meanings or referents of the number words are collections of entities" (Fuson, 1990, p. 273). For example, a child who disembeds understands that 43 is composed of 40 or 4 tens (groups of 10 objects) and 3 single objects. When children learn to recite the decades in sequence (e.g., 10, 20, 30, 40), they initially interpret each decade as a grouping of ten. Thus, in the example above, a child focuses on the objects in the groups and has a count of 40 so far, with 3 more loose objects. This sequential count of the tens or decades becomes linked to the groups of ten and, in time, the children may count the groups as *units of ten* (e.g., "1 ten, 2 tens, 3 tens, 4 tens" or "1, 2, 3, 4, . . . 4 tens"). These units of ten are initially separate from any remaining *units of one*, but eventually the units of ten and the units of one are inte-

grated. Then the child has great flexibility both for counting and problem solving as interpretations of 43 rapidly shift from 43 separate objects (1, 2, 3, . . . , 43 *ones*), to 4 groups of ten objects each (10, 20, 30, 40 *ones*) and 3 more objects (3 *ones*), to 4 groups of ten objects (1, 2, 3, 4 *tens*, so that's 40) and another set of 3 more objects (3 *ones*), to knowing that 4 tens and 3 ones is *43* (Fuson, Wearne, et al., 1997). Understanding multiunit structures is the foundation for developing place value concepts.

The complexity of interpreting multidigit numbers is influenced by language (Fuson & Kwon, 1992; Yang & Cobb, 1995). The English system of number words does not support the conceptual components of multiunit numbers. There are irregularities in the English system that make it difficult to count and to identify the tens and ones structure in two-digit numbers. For example, as children learn the counting word sequence, there is consistency in counting *within* decades (e.g., 1, 2, 3, . . . 7, 8, 9; 21, 22, 23, . . . 27, 28, 29), but children have difficulty bridging *across* decades (e.g., 21, 22, . . . 28, 29, ??). In terms of supporting conceptual structures, 26 is named "twenty-six" in the English language. The named value representation for 26 in several Asian languages (e.g., Chinese, Japanese, Korean), is "two ten six," making explicit the idea that the number is composed of 2 units of ten and 6 units of one. In English the link between the number word for numbers between 10 and 20 and the conceptual structure of the number is even less transparent. For example, in the number word "thirteen" there is little support for children's construction of 13 as 1 unit of ten and 3 units of ones. In a comparison of the children who spoke English or an Asian language, Fuson and Kwon (1992) reported that speakers of the Asian languages seemed to construct multiunit conceptual structures for two-digit numbers earlier than the English-speaking cohort. The Asian languages, which emphasize the conceptual structure of numbers, were found to influence children's understanding of multiunits. Essentially, the string of English counting words implies a unitary meaning for numbers and does not encourage children to think of number with the more sophisticated construction of multiunits.

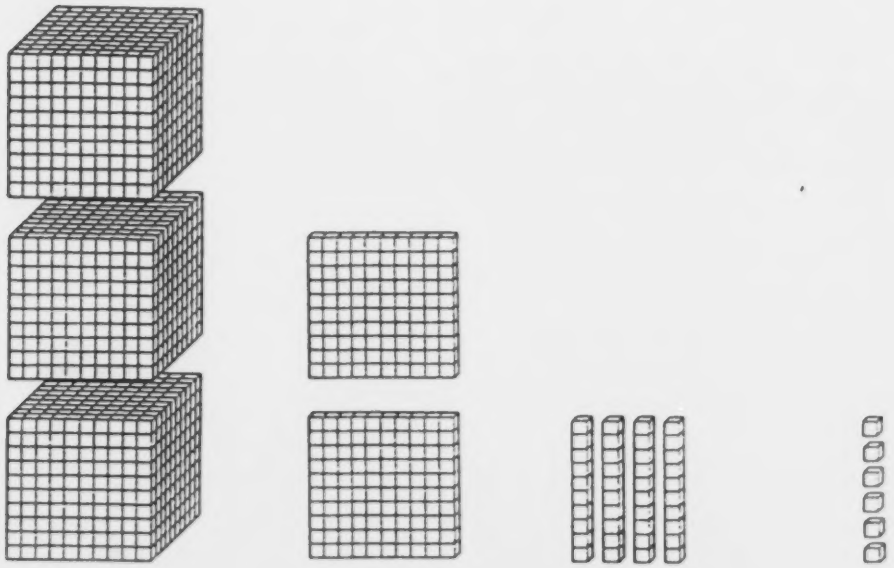
The transition from counting by ones to counting by tens, hundreds, and so on is necessary in order for children to develop effective problem-solving strategies and meaning for place value. Successful manipulation of symbols within computation problems does not imply this transition. Children may compute accurately without understanding (Kamii & Joseph, 1988). Several tasks have been developed to assess children's understanding of number and numeration. For example, Kamii (1989) describes a digit correspondence task where a child is shown the numeral

16 and asked to construct a set with sixteen chips. Then the child is asked which part of the set is represented by each of the digits in the numeral, first focusing on the 6, then on the 1. Children who do not understand place value often correctly respond that the 6 represents six chips in the set, but they assign only one chip, rather than ten chips, to the numeral 1, the tens digit. This response indicates that the multidigit number is perceived of as a series of concatenated single digits and the relative position of the digits is not significant. As a result of this assessment, Kamii (1989) posits that direct instruction with traditional addition and subtraction algorithms, where columns of number are treated as single units and operated on from right to left, inhibits children's understanding.

Place-Value Instruction. There is continuing discussion (Baroody, 1990; Fuson, 1990) as to whether children should first establish a firm understanding of base-ten numbers with place-value concepts before investigating multidigit addition and subtraction procedures. Some researchers feel that the learning of place-value concepts and multidigit procedures should be integrated (Fuson, Smith, & Lo Cicero, 1997). Hiebert and Wearne (1996) noted the interrelationship of place-value understandings and use of multidigit procedures for addition and subtraction. However, their study did not clarify whether procedural strategies developed after or concurrent with place-value understandings.

Whether multidigit addition and subtraction are taught after or concurrent with place value, understanding how children make the transition from unitary to multiunit conceptual structures has direct implications for instruction. For example, Fuson and Briars (1990) reported a series of studies in which first- and second-grade teachers used base-10 blocks (see Figure 5.1) to support children's construction of multiunit conceptual structures and an understanding of place value concepts. This structured teaching/learning setting was characterized by the study of advanced counting strategies for adding and subtracting sums to 18 followed by work with groupings of ten. Children were then taught how to link actions on the base-10 blocks with written procedures for adding and subtracting multidigit numbers; trading ten-for-one was always ordered from right to left. The use of base-ten blocks seemed to support the construction of multiunits for second-graders and above-average first-graders. Average and below-average first-graders maintained a unitary counting sequence, but they did not construct multiunit conceptual structures. When problem-solving instruction is carefully designed to promote classroom discourse and to facilitate the link between children's emerging interpretations of multidigit numbers, their actions on concrete models,

Figure 5.1. Base-ten blocks.



and symbolic representations, children's understanding of place value and multidigit addition and subtraction increases (Hiebert &Wearne, 1993).

In general, the use of concrete models to help facilitate young children's mathematical thinking has been accepted as an appropriate teaching strategy. However, the use of manipulatives, such as base-ten blocks or Unifix cubes, to help children make the transition from constructing unitary to multiunit conceptual structures raises two specific issues (Baroody, 1989; Kamii, 1989).

First, if manipulatives are used in instruction, there is a caution about what children are actually learning from them. Ross (1989) suggests that

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concrete embodiments have their limitations and may be successfully manipulated without meaning. A manipulative can be treated as a symbol system. Children may learn the rules and procedures associated with a given manipulative material without ever establishing the link between that manipulative and the concept it is to embody. For example, the relationship of ten-for-one already exists in base-ten blocks where 1 ten is a rod of 10 units and 1 hundred is a flat of 10 tens. With the blocks, children can practice trading 10 units for 1 ten and 10 tens for 1 hundred. They can successfully group a set of 26 units and trade the grouped units for 2 tens. However, when asked to count the resulting collection of 2 tens and 6 ones, a young child who has not mentally constructed ten as a unit will proceed to count the entire collection by ones or count the first ten as a unit and then continue counting by ones, "10, 11, 12, 13, . . . 26." A child who has mentally constructed ten as a unit will count "10, 20, 21, 22, 23, 24, 25, 26." Unless young children are asked to communicate their thinking, it is not clear what they understand about a task involving concrete models or what mathematical concept they have constructed.

The second issue regarding manipulatives is the question of relevance. Kamii (1989) asserts that requiring children to physically construct sets of ten with manipulatives is not helpful because construction of multiple groupings can only be an intellectual activity; she holds that place value cannot be perceived from concrete materials.

Young children need to engage in a variety of counting tasks where quantity is partitioned into parts and the parts are subsequently related to the whole quantity. In kindergarten and first grade, children may investigate how many ways a set of objects (e.g., 9 chips) may be separated into two or more parts (e.g., three sets of 3 chips; 5 chips and 4 more chips). In first and second grade, this partitioning supports place-value construction as children understand, for example, 34 as 3 sets of ten and 4 more. This flexibility with numbers is evident when a child who solves a problem such as 27 + 35 states: "20 plus 30 is 50 . . . 7 more is 57 . . . and 5 more, . . . 58, 59, 60, 61, 62." This solution strategy clearly demonstrates a mature understanding of number. The power of the part–whole construct for number is its flexibility. In a curriculum that encourages the flexible use of the part–whole construct, addition and subtraction problems are open to many solution strategies, limited only by the numerical insights of the child. For example, another child may solve this same problem by counting: "35, . . . [+10] 45, . . . [+10] 55, . . . [+7], 56, 57, 58, 59, 60, 61, 62."

Both of these solution strategies are examples of informal algorithms or invented strategies that children may construct, building on their understanding of multidigit numbers and their ability to coordinate the

counting of tens and ones. Informal strategies are invented both individually and collectively as children use manipulatives to model quantity and numerical relationships and as children use counting strategies. Frequently, children's invented strategies seem to be an application of place-value knowledge. There is growing evidence that when arithmetic concepts and procedures are approached as problem solving, children's invented strategies foster number sense and understanding of multidigit addition and subtraction (Carpenter et al., 1998). However, it is critical that teachers hold children's informal algorithms to mathematical standards for what is mathematically correct, generalizable, and efficient. Further, teachers and curriculum developers cannot presume that every response that a child offers will be correct, complete, or easily interpreted, just because the child can explain and justify his or her strategy (Campbell, 1997).

Development of Problem-Solving Abilities

During the past 15 years, studies on children's thinking have yielded a well-developed body of research defining addition and subtraction word problems as well as the strategies young children use to solve those problems (Briars & Larkin, 1984; Carpenter & Moser, 1983, 1984; Carpenter, Moser, & Romberg, 1982; Riley et al., 1983). Addition and subtraction problems have been classified according to the action or relationship of the quantities in the word problem (Carpenter & Moser, 1983; Riley et al., 1983). The relative difficulty of these problems is determined by children's solution strategies.

Research indicates that prior to formal instruction, children can successfully solve simple word problems by directly modeling the action in the problem with concrete objects and then counting the objects. Young children then develop more advanced strategies for solving problems, moving from modeling, to advanced counting strategies, to use of derived facts, to the recall of number facts. Even young children who generally use direct modeling to solve problems may use derived facts or recall strategies occasionally, depending on the problem type and the size of the numbers involved. For example, consider this problem: *Anna had some balloons. She gave 7 of them to her best friend. Now Anna has 5 balloons left. How many balloons did Anna have to start with?* Children who use a derived fact to solve this problem may recognize that $7 = \underline{5} + 2$ and thus $5 + \underline{5} = 10$ and $10 + 2 = \underline{12}$, the solution. Children become more flexible in their choice of solution strategy as a result of changes in their conceptual knowledge (Carpenter, 1985).

Young children's successful problem-solving abilities are well docu-

mented. However, children's problem-solving experiences in traditional classrooms are limited (Stigler, Fuson, Ham, & Kim, 1986). In most programs children are restricted to simple word problems where two sets are joined or some amount is removed from a set. Because these types of problems do not challenge children's thinking, particularly in terms of the solution strategies required to solve them, the current early childhood curriculum can be considered impoverished. Children should be given a variety of word problems to solve. Further, the implication of this research for curriculum development is clear: "It is not necessary to defer instruction on word problems until computational skills have been mastered" (Romberg & Carpenter, 1986, p. 855).

The decision about what problems to consider should be determined by the teacher's knowledge of the relative difficulty of the problem types and knowledge of individual children's solution strategies. The critical curriculum issue is not whether the problem can be classified as addition or subtraction. For example, consider the following problem: *Keisha saved 6 pennies. How many more pennies does she need to save to have 10 pennies altogether?* This type of problem has traditionally been referred to as a missing addend problem. This problem, once determined by some to be too difficult for first-grade children, can be solved by subtracting 6 from 10. This subtractive solution strategy requires an explicit understanding of the part–whole relationship. However, first-grade children are very successful at solving this problem using an appropriate additive solution strategy (e.g., 6, . . . 7, 8, 9, 10, . . . 4 pennies) (Briars & Larkin, 1984; Carpenter & Moser, 1984; De Corte & Verschaffel, 1987; Riley et al., 1983). Solving the symbolic missing addend problem, 6 + ____ = 10, without the support of a problem context is a difficult task for young children. The experience of solving meaningful word problems that can be represented by a missing addend sentence should not be confused with having children solve symbolic computation problems with missing addends. These are two distinct tasks that require different knowledge.

Young children also can solve simple multiplication (grouping) and division (partitioning) word problems by direct modeling (Kouba, 1989). Indeed, even kindergarten-aged children can solve multiplicative problems if concrete materials are present and if the actions or relationships described in the problems can be modeled by movement of counters (Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993). The development of children's solution strategies for these problems is similar to those for addition and subtraction. Direct modeling is replaced with counting strategies and eventually by derived facts and recall. Children do not suddenly advance from one strategy to the next; rather they seem to develop an increasing number of alternative approaches (Carpenter et al., 1993; Mul-

ligan & Mitchelmore, 1997). Because young children do not view multiplication as commutative, modeling 3 sets of five is a different task from modeling 5 sets of three. As a result of this distinction, children also perceive division problems differently. In a problem such as, *Max has 12 marbles. He wants to put 4 marbles in each bag. How many bags will he need?*, children construct a set of 12 objects and then make groups of four. When all the objects have been grouped, the number of groups is counted for the solution. In a problem such as, *Max has 12 marbles and 3 bags. If he puts the same number of marbles in each bag, how many marbles are in each bag?*, children construct a set of 12 objects and then deal them out into three piles until all the objects are dealt. The number of objects in a pile is counted for the solution. Both solution strategies directly model the problem situation; therefore children's actions on the objects are different for the two problems. These problems are challenging, yet they are appropriate in the early childhood curriculum because they can be modeled meaningfully by young children and they build on children's understanding of addition and subtraction. The key is to allow the children to solve them in their own ways, as opposed to requiring assignment of a recalled multiplication fact (Clark & Kamii, 1996).

Curriculum decisions about what types of word problems to include in early childhood mathematics should be made by considering whether the problem is set in a meaningful context for children, portraying appropriate mathematical structures. Young children solve problems that go beyond simple addition and subtraction events.

Representation

Young children understand number and can solve problems without using written symbols. Children are introduced to the use of symbols to represent mathematical concepts so they may develop meaning for written symbolic representations. The process of connecting written symbols with real-world referents, such as objects or actions on objects, develops meaning for symbols (Hiebert, 1988). The links young children have established between a symbol and its referent need to be transparent before young children can be expected to manipulate symbols as abstract quantities.

Number sentences (e.g., $a + b =$ ____, $a - b =$ ____) are symbolic representations used by young children to directly model the action described in problems. Number sentences are an extension of children's problem-solving abilities and reflect their solution strategies. As children become more flexible in their choice of solution strategy, they are able to use a variety of appropriate number sentences. Consider the problem:

Casey had some pennies. She lost 5 of them. Now she has 7 pennies left. How many pennies did Casey have to start with? First-grade children link the representation ____ −5 = 7 with this problem when given a choice of several alternative number sentences (Carey, 1991). As children develop a more generalized part–whole schema, they also use 7 + 5 = ____ as an appropriate representation for this problem. Adding the two parts given in the problem is an explicit representation of the part–whole relationship. This is a more sophisticated representation because it does not directly translate from the problem situation. Although children can solve problems, there are differences in the number sentences used to represent them. Decisions about what representations are appropriate are determined by how young children think about a problem.

Geometric Concepts and Spatial Sense

> Mathematics curricula at all levels must introduce more of the breadth and power of the mathematical sciences. As mathematics is more than calculation, so education in mathematics must be more than mastery of arithmetic. Geometry . . . [is] as important as numbers in achieving mathematical power. . . . To prepare students to use mathematics in the twenty-first century, today's curriculum must evoke the full spectrum of the mathematical sciences. (National Research Council, 1989, p. 43)

The traditional mathematics curriculum for young children has been dominated by the study of number and numerical relationships. One study noted that *less than 7%* of the pages in basal mathematics textbooks for kindergarten through fourth grade conveyed geometry (Fuys, Geddes, & Tischler, 1988). In the United States, there is no coherent, accepted geometry curriculum for children (Usiskin, 1987).

If mathematics is perceived as the search for order, pattern, and relationships to characterize ideas and experiences, then geometry and spatial sense should be central topics in a mathematics curriculum for young children. Through hands-on experiences with the geometric nature of their world, young children may construct geometric ideas. The traditional geometry curriculum for young children has emphasized vocabulary, focusing on verbal definitions and rules. Current research indicates that children in the upper primary grades generally do not use these "learned" definitions and rules when they solve problems involving geometry and space. Instead, they rely on the conceptual structures they have constructed from real-life experiences, both in school and at home. This reliance on informally constructed geometric concepts as opposed to verbal definitions has been noted not only for children who have incom-

plete recall of definitions and rules as presented in school, but also for children who can accurately restate the verbal rules and definitions (Clements & Battista, 1989, 1992).

Research indicates that young children can discern some of the properties of shapes and can readily think of two-dimensional figures in terms of paths and motions (Clements & Battista, 1989). Instruction can foster young children's geometric understandings and improve their spatial-perception abilities, but the instruction should start at a young age and provoke children to reflect on the manipulation of geometric structures and to construct concepts interpreting perceived geometric properties of figures (Clements & Battista, 1992; Del Grande, 1987). Research suggests a framework for developing curriculum and instruction in geometry (van Hiele, 1986). For example, young children should manipulate concrete geometric shapes and consider geometric forms in real-world settings. This reliance on visual interpretation is hypothesized as a mechanism for fostering the child's construction of definitions and recognition of geometric relationships (Fuys et al., 1988).

Although calls for change are clear, incorporation of geometry and spatial sense in school mathematics remains an important yet prodigious charge. With that challenge in mind, some researchers have begun to develop curriculum units to support geometric knowledge and spatial sense (e. g., Clements, Battista, Sarama, Swaminathan, & McMillen, 1997).

CREATING PROBLEM-SOLVING ENVIRONMENTS

The Role of the Teacher

The early childhood mathematics curriculum is influenced by teachers' instructional decisions. Decisions should be based on what is known about the development of children's thinking and the thinking evidenced by individual children in the classroom. Mathematics instruction facilitates a child's construction of knowledge when a teacher knows what a child understands about a concept and how that concept develops. Teachers' knowledge of their children's problem-solving skills is related to student learning (Carpenter et al., 1989; Fennema, Franke, Carpenter, & Carey, 1993). The assessment of problem-solving skills need not be accomplished in a formal task; it can occur during instruction. When teachers observe children's approaches to problems, when they listen to children's explanations, when they challenge children to clarify their thinking, and when they evaluate the level of sophistication inherent in

children's individual solutions, teachers are assessing. Assessing children's thinking is a continuing process and an important part of effective instruction. Appropriate assessment and instructional tasks can be determined as a result of teachers' questioning, listening to, and observing children. This type of ongoing instruction/assessment provides a natural context for children to do and to communicate mathematics. Communication in classrooms serves two purposes: (1) It fosters children's development of patterns of verbal communication as they talk about mathematics, and (2) it allows teachers to learn about their children's thinking. When teachers ask questions, they focus children's attention on a particular aspect of a task or they convey expectations. Instructional tasks and classroom discourse affect how children perceive a task, how they process information, and what relationships they construct. When children have increased opportunities to reflect and to explain their thinking, increased achievement results (Hiebert & Wearne, 1993).

Learning Contexts

Curriculum focusing on mathematical ideas and problem solving needs resources for realistic, engaging problems. The best resources are the experiences of the children, either planned experiences occurring in the school setting or experiences from outside the classroom that are common to all the children. The critical issue is to conceive of problems that are meaningful to *each* child and are part of *every* child's environment. Then the mathematics classroom may be a place "where ideas are accepted, where suggestions are investigated, and where meaningful problems are solved" (Campbell & Langrall, 1993, p. 110). For this to occur, preschool and primary classrooms must be places where each child's thinking is valued, where the involvement of each child is expected, where sufficient time is permitted for investigations, and where questioning and listening are the norm (Campbell & Langrall, 1993). When this occurs, the classroom becomes a powerful force in a young child's life. It is the place where "knowledge is continuously recreated, recycled and shared" (Ladson-Billings, 1992, pp. 113–114).

CHALLENGE FOR THE FUTURE

This is a vibrant period for mathematics education, as educators from all avenues (mathematics educators, mathematicians, practitioners, and policy makers) seek to reform and revitalize mathematics curriculum and

instruction. The premise of this chapter is that these changes in mathematics education must begin in the early childhood years.

When mathematics is perceived as meaningful, understandable, and challenging, a child experiences the power of mathematics.

> Mathematics . . . provide[s] one of the few disciplines in which the growing student can, by exercising only the power inherent in his or her own mind, reach conclusions with full assurance. More than most other school subjects, mathematics offers special opportunities for children to learn the power of thought as distinct from the power of authority. This is a very important lesson to learn, an essential step in the emergence of independent thinking. (National Research Council, 1989, p. 4)

This chapter presents a review of research that may direct and guide curriculum reform to meet the challenge implied above. An early childhood mathematics "curriculum should include a broad range of content, . . . emphasize the application of mathematics, . . . be conceptually oriented, . . . actively involve children in doing mathematics, . . . [and] emphasize the development of children's mathematical thinking and reasoning abilities" (Commission on Standards for School Mathematics, 1989, pp. 18–19).

Acknowledgment. The work reported herein was supported by grants from the National Science Foundation to the author (ESI 94-54187 and ESI 95-54186). Any opinions, findings, and conclusions expressed in this publication are those of the author, and no endorsement from the National Science Foundation should be inferred.

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